Fourier Transforms

Born: 21 March, 1768 (Auxerre, Bourgogne, France); Died: 16 May 1830 in Paris, France

HartRAO

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Acknowledgements

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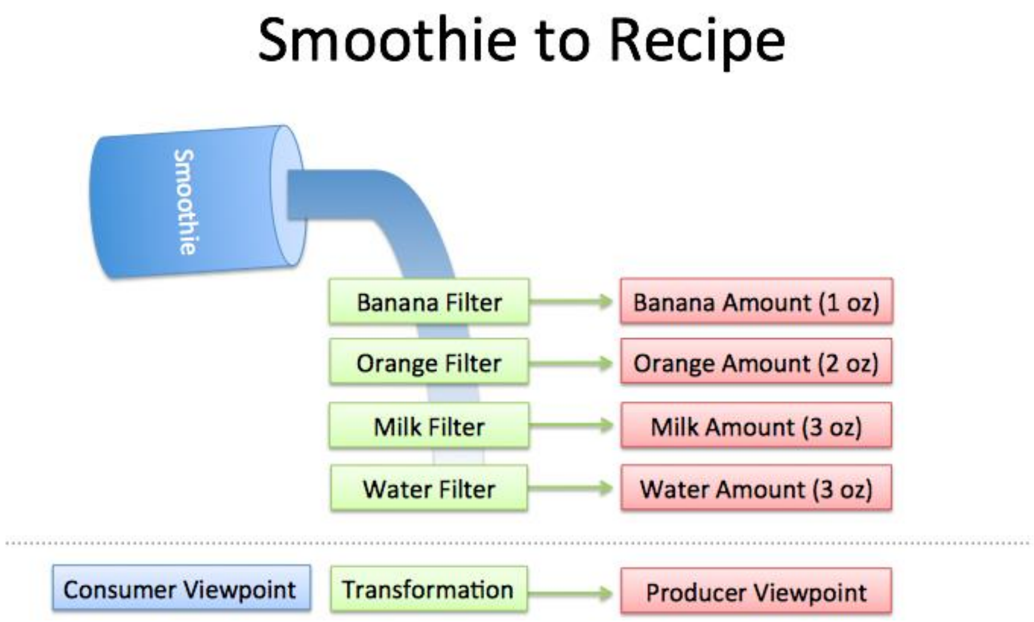
What is the Fourier Transform?

- Wikipedia: a Fourier Transform decomposes a function of time into the frequencies that make it up
- In this case, the Fourier transform transforms between the ***time*** and the ***frequency*** domains
- A good example is sound:
- Listen to a tone – say the note A at 440 Hz. You don't feel your ear getting hit 440 times a second, you just hear a single tone. The air molecules hitting your ear 440 times a second is the ***time***-domain representation while the single tone you hear is the ***frequency***-domain representation of the same signal.
- Note that time and frequency have reciprocal units, e.g. unit of time = 1 sec, unit of frequency = 1 sec^{-1} = 1 Hz
- Fourier transform is a transform to a reciprocal space, so will always transform from units of x to units of x^{-1}

What Does the Fourier Transform Do?

- **What does the Fourier Transform do?** Given a smoothie, it finds the recipe.
- **How?** Run the smoothie through filters to extract each ingredient.
- **Why?** Recipes are easier to analyze, compare, and modify than the smoothie itself.
- **How do we get the smoothie back?** Blend the ingredients again.



credit: Kalid Azad

What is the Fourier Transform?

- Fourier transform is
  - reversible
  - linear
- For any function $f(x)$ (which in astronomy is usually real-valued, but $f(x)$ may be complex), the Fourier transform can be denoted $F(s)$, where the product of x and s is dimensionless. Often x is a measure of time t (i.e., the *time-domain* signal) and so s corresponds to inverse time, or frequency (i.e., the *frequency-domain* signal).
- Fourier transforms are important because they often transform a complicated problem in one domain to a much simpler problem in another domain.

Expansion of a Function in a Series

- Taylor-series expansion of a function

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

Fourier Series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Fourier series make use of the orthogonality relationships of the sine and cosine functions

Fourier showed that any continuous function $f(x)$, can be expressed as a Fourier series. The Fourier transform is just generalization of the above series where sums are replaced by integrals

What is a Fourier Transform?

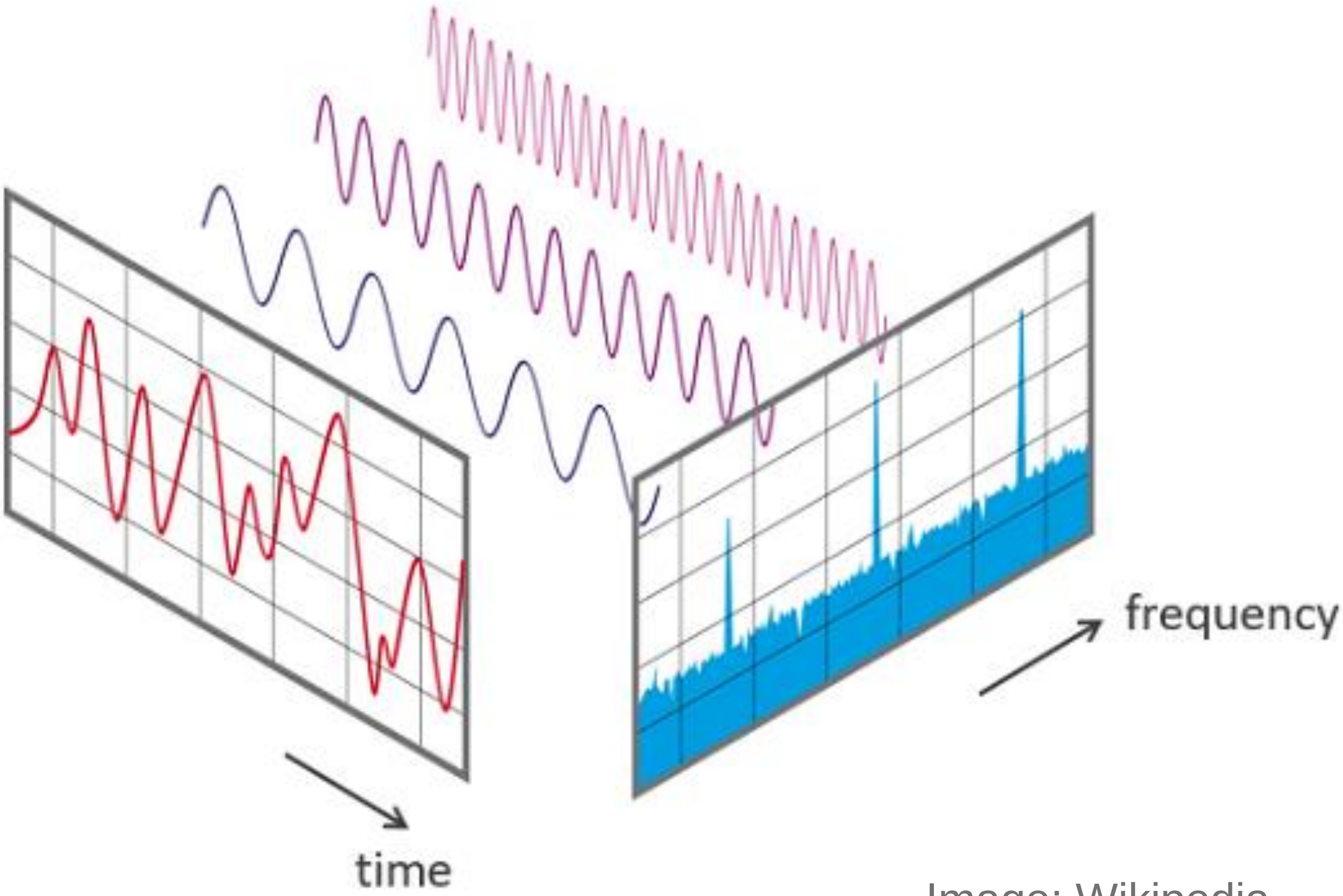
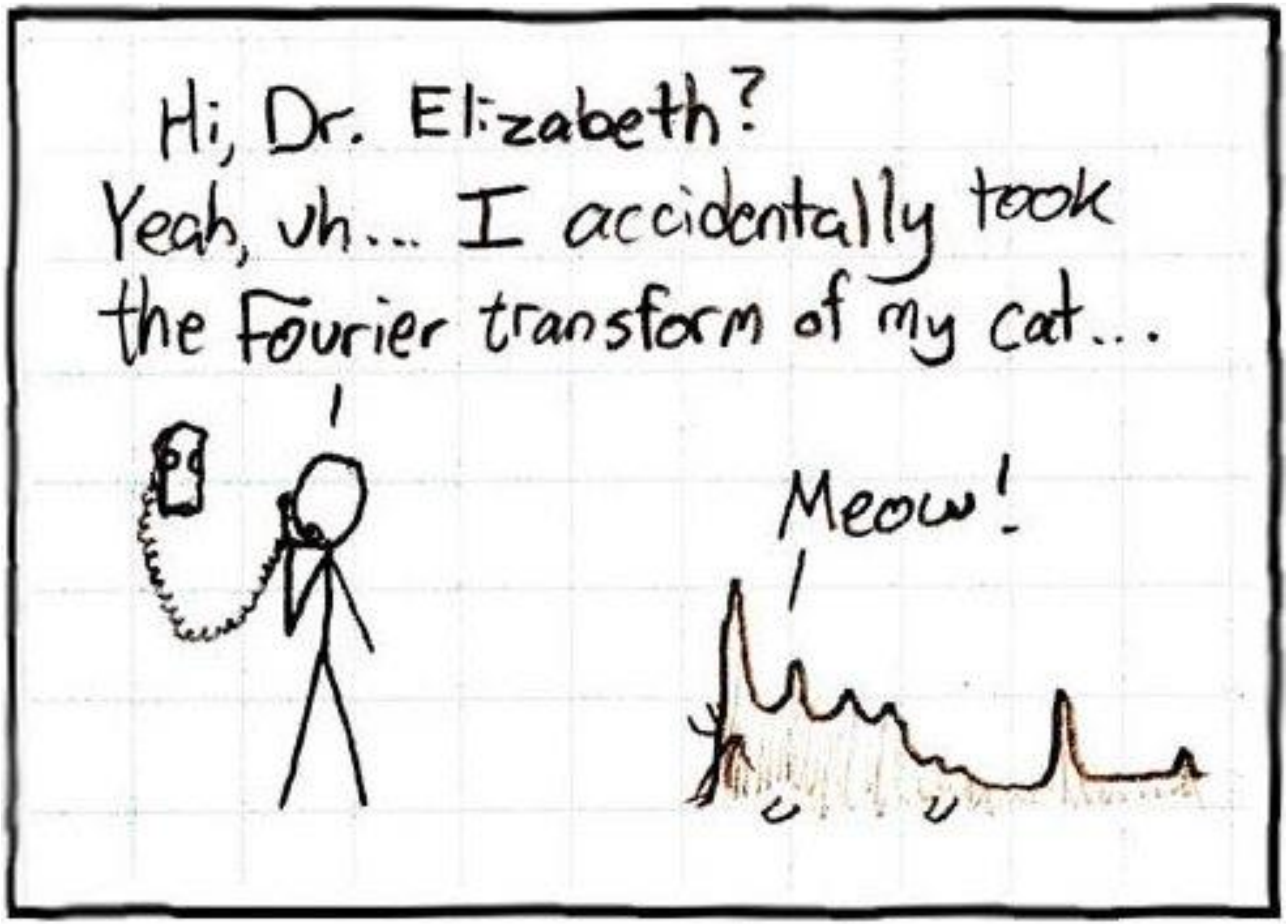


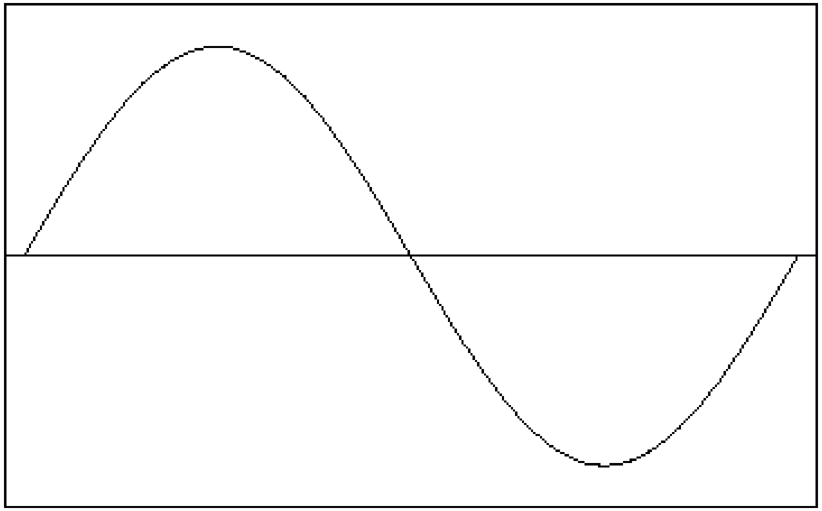
Image: Wikipedia



XKCD **https://xkcd.com/**

Example: Square Wave

- Animation of a square wave



Credit:Dr. Dan Russell, Grad. Prog. Acoustics, Penn State

- As more and more Fourier terms or sine waves are added, the shape more and more closely approaches a square wave

Not Just Time and Frequency

- The Fourier transform is equally applicable in other areas to transform between two different representations
- It can easily be generalized to more than one dimension. For example: two dimensions, so it could be used for two dimensional functions, such as an image which can be represented as brightness(x,y)
- In interferometry it is mostly used to transform between the spatial domain – an image (of the sky) – to the spatial frequency domain.

One-Dimensional Fourier Transform

- The one-dimensional Fourier transform and its inverse
- Fourier transform (continuous case)

$$F(s) \equiv \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i s x}\, dx \ ,$$

Where $i = \sqrt{-1}$ in both cases

- Inverse Fourier transform:

$$f(x) \equiv \int_{-\infty}^{\infty} F(s)\, e^{2\pi i s x}\, ds \ ,$$

- Alternative definitions of the Fourier transform are based on angular frequency ($=2\pi\nu$), have different normalizations, or the opposite sign convention in the complex exponential. Since Fourier transformation is reversible, the symmetric symbol $\Leftrightarrow$ is often used to mean "is the Fourier transform of"; e.g., $F(s) \Leftrightarrow f(x)$.

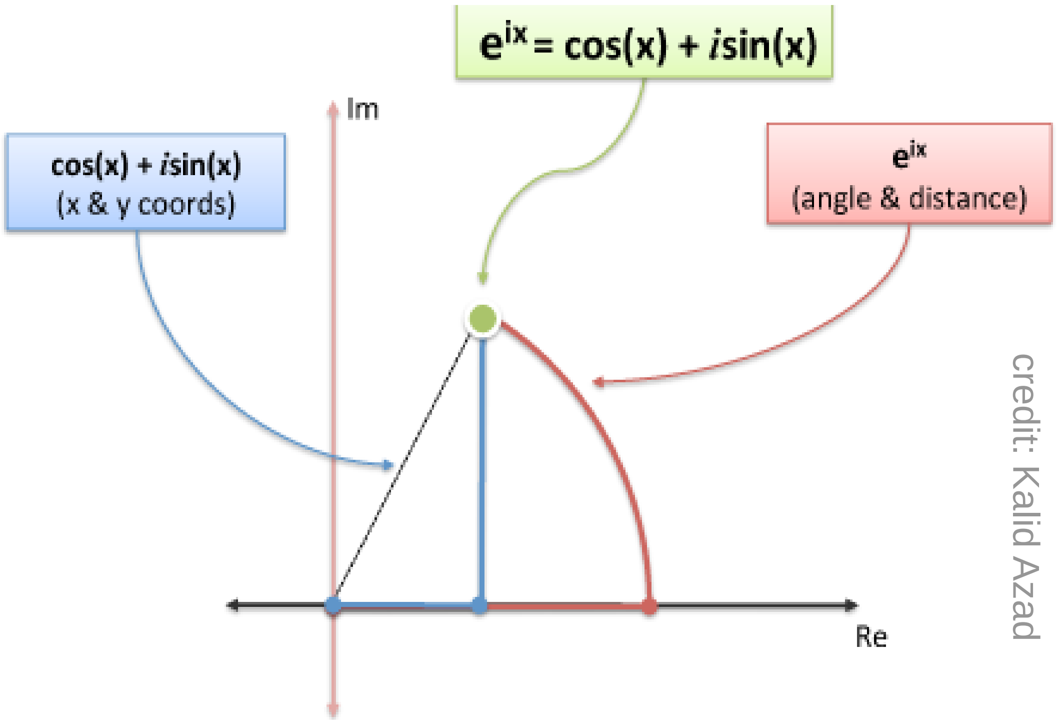
Complex Exponential

- The heart of the Fourier transform is the complex exponential.
- A complex exponential is simply a complex number where both the real and imaginary parts are sinusoids. The exact relation is called ***Euler's formula***

$$e^{i\phi} = \cos\phi + i\sin\phi,$$

Where $i = \sqrt{-1}$

- Complex exponentials are much easier to manipulate than trigonometric functions, and they provide a compact notation for dealing with sinusoids of arbitrary phase, which form the basis of the Fourier transform.



credit: Kalid Azad

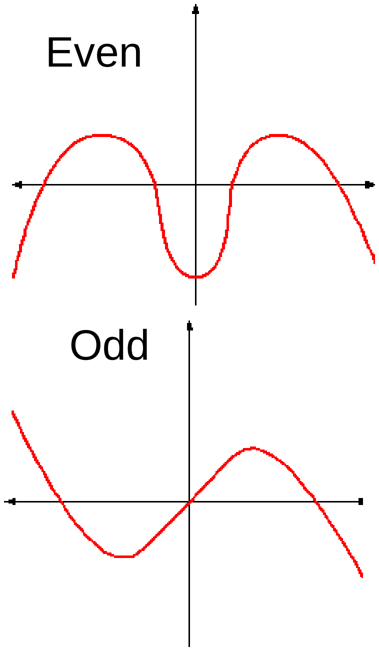
Discrete Fourier Transforms (DFT)

- The result of the DFT of an N-point input time series is an N-point frequency spectrum, with Fourier frequencies k ranging from $-(N\text{-}1)/2$, through the 0-frequency or so-called "DC" component, and up to the highest Fourier frequency $(N\text{-}1)/2$.
- Each bin number represents the integer number of sinusoidal periods present in the time series.
- The amplitudes and phases represent the amplitudes A_k and phases, Φ_k, *of* those sinusoids.
- Each bin can be described by $X_k = A_k e^{-i\phi_k}$
- For real-valued input data, however, the resulting DFT is Hermitian—the real-part of the spectrum is an even function and the imaginary part is odd, such that $X_{-k} = \overline{X}_k$, where the bar represents complex conjugation.
- All of the "negative" Fourier frequencies provide no new information, so the total number of independent pieces of information (i.e. real and complex parts) is N, just as for the input time series. No information is created or destroyed by the DFT.
- Usually computed by Fast Fourier Transform (FFT) algorithm: **requires** regularly sampled data.

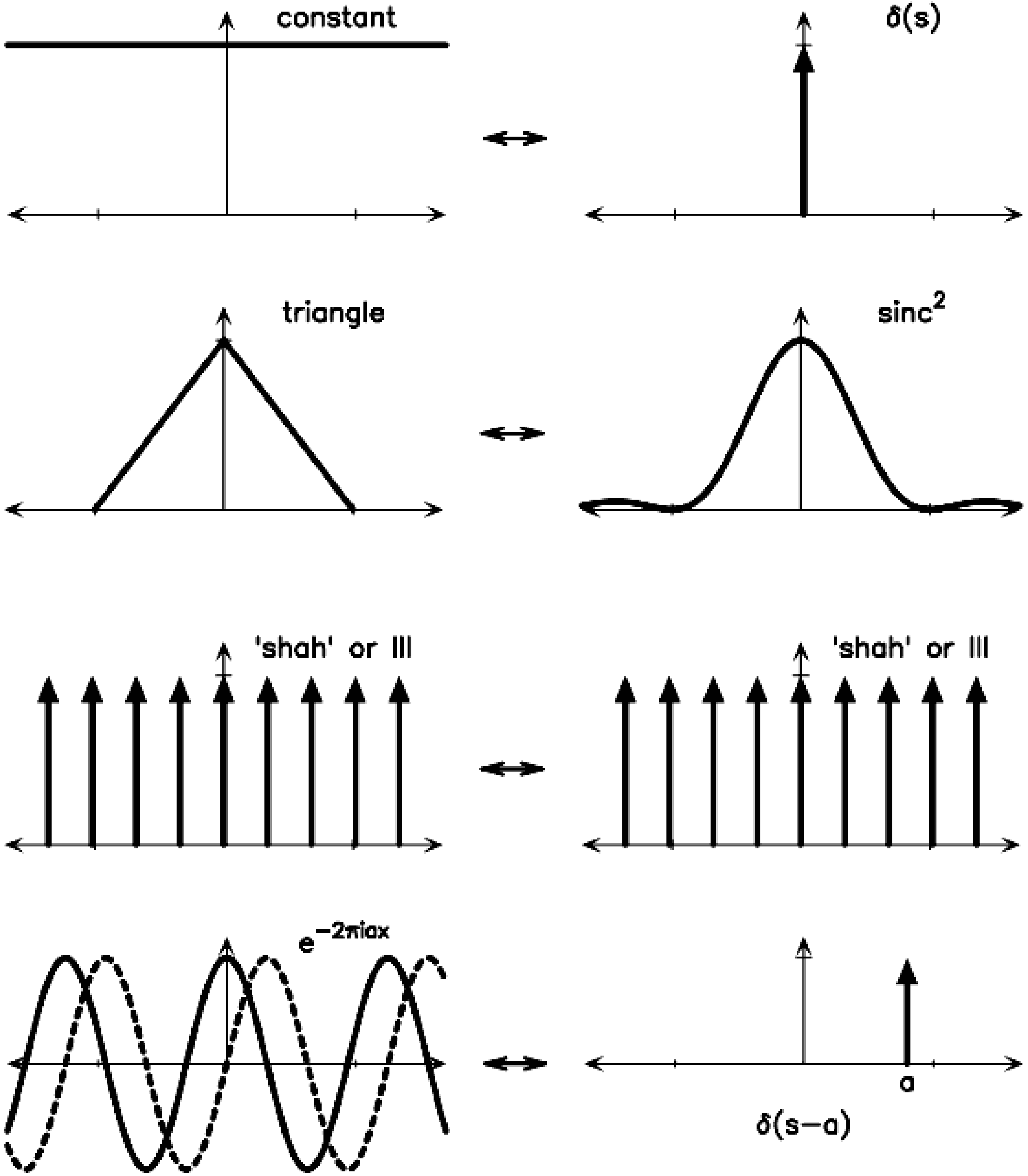
Symmetries

- There are a number of useful symmetries in functions and their Fourier transforms:

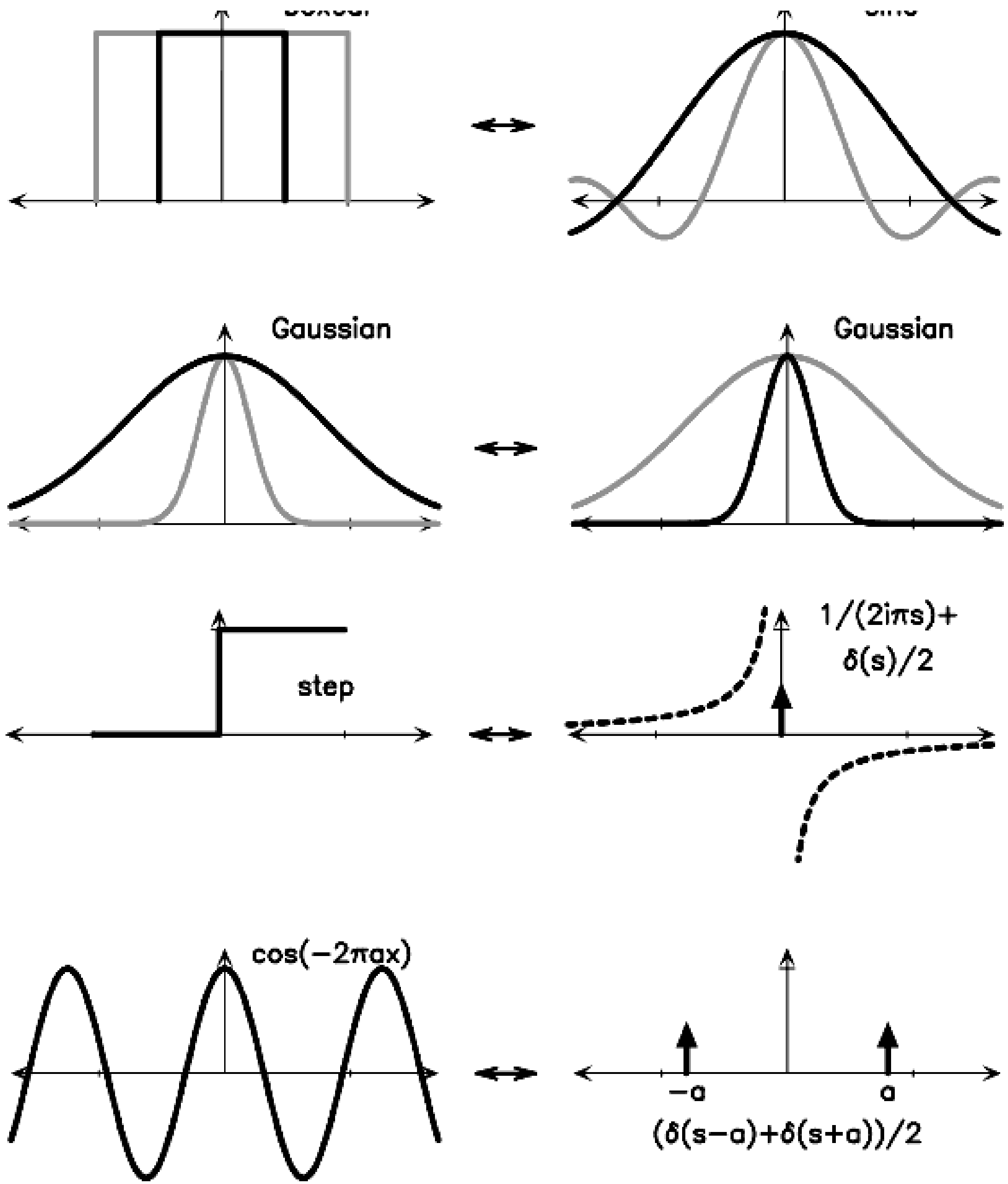
function *g*(*x*), e.g. time domain	**Fourier transform, *G*(*s*), e.g. frequency Domain**
real	Hermitian (real=even, imag.=odd)
imaginary	anti-Hermitian (real=odd, imag.=even)
even	even
odd	odd
real & even	real & even (i.e. cosine transform)
real & odd	real & odd (i.e. sine transform)
Imaginary & even	imaginary & even
imaginary & odd	real & odd



Basic Transforms I



Basic Transforms II



Fourier Theorems I

- **Addition Theorem**: The Fourier transform of the addition of two functions $f(x)$ and $g(x)$ is the addition of their Fourier transforms $F(s)$ and $G(s)$. This basic theorem results from the linearity of the Fourier transform. A special case of the addition theorem states that if a is a constant, then $af(x) \Leftrightarrow aF(s)$

$$f(x) + g(x) \Leftrightarrow F(s) + G(s)$$

- Shift Theorem: a function $f(x)$ shifted along the x-axis by a to become $f(x-a)$ has the Fourier transform $e^{-2\pi ia}F(s)$. The magnitude of the transform is the same, only the phases change.

$$f(x-a) \Leftrightarrow e^{-2\pi ias}F(s)$$

Fourier Theorems II

- **Similarity Theorem**: For a function *f(x)* with a Fourier tranform *F(s)*, if the *x*-axis is scaled by a constant a so that we have *f(ax)*, the Fourier transform becomes $|a|^{-1} F(s/a)$. In other words, a "wide" function in the time-domain is a "narrow" function in the frequency-domain. This is the basis of the uncertainty principle in quantum mechanics and the diffraction limits of radio telescopes.

$$f(ax) \Leftrightarrow \frac{F(s/a)}{|a|}$$

- **Modulation Theorem**: The Fourier transform of a function *f(x)* multiplied by $\cos(2\pi\nu x)$ is $\frac{1}{2}F(s-\nu) + \frac{1}{2}F(s+\nu)$. This theorem is very important in radio astronomy as it describes how signals can be "mixed" to different intermediate frequencies (i.e. IFs).

$$f(x)\cos(2\pi\nu x) \Leftrightarrow \frac{1}{2}F(s-\nu) + \frac{1}{2}F(s+\nu).$$

Fourier Theorems III

- **Derivative Theorem**: The Fourier transform of the derivative $f'(x)$, of a function $f(x)$, is $i2\pi sF(s)$

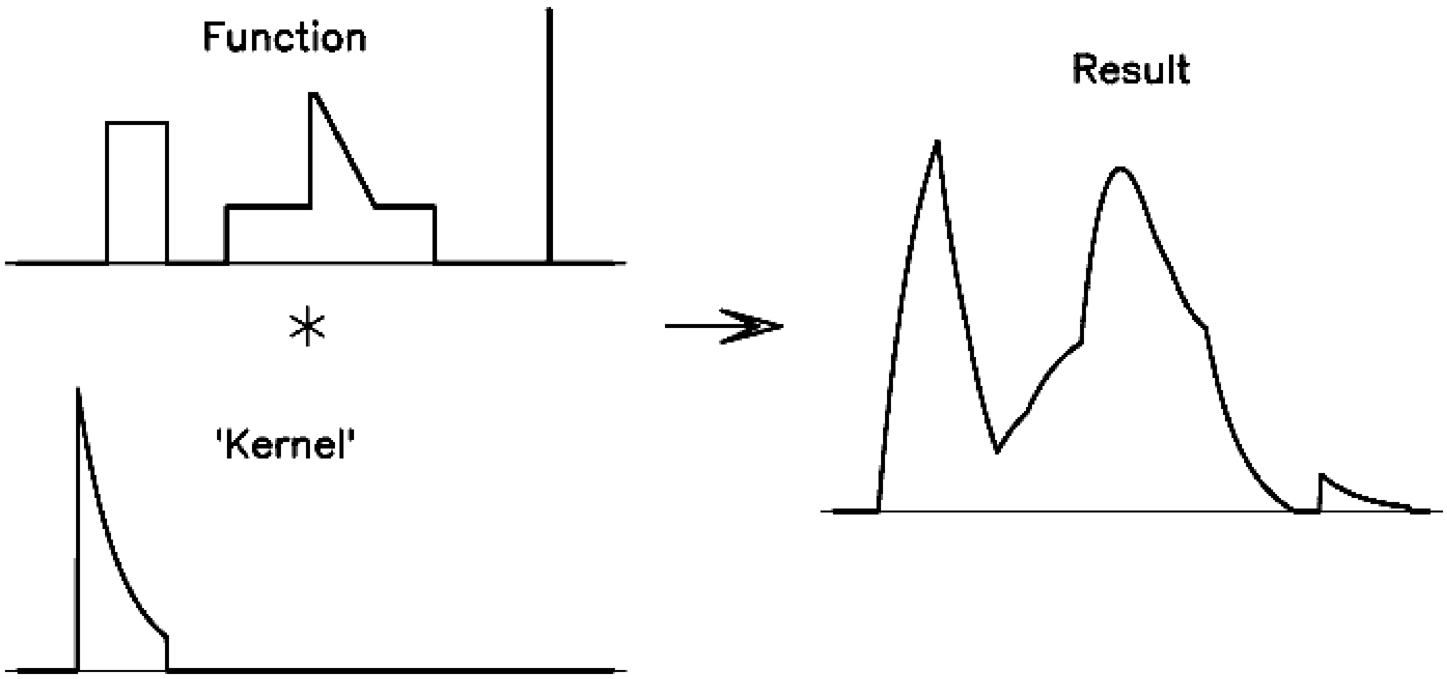
$$f'(x) \Leftrightarrow i2\pi sF(s)$$

Convolution & Cross-Correlation

- Convolution shows up in many aspects of astronomy, most notably in the point-source response of an imaging system and in interpolation.
- We will represent convolution, by $*$ (the symbol $\otimes$ is also frequently used)
- multiplies one function, f, by the time-reversed function g, shifted by some amount x, and integrates from -∞ to +∞.
- The **convolution**, $h(x)$, of the functions $f(x)$ and $g(x)$ and is a linear function of f and g defined by:

$$h(x) = f * g \equiv \int_{-\infty}^{\infty} f(u) g(x-u)\, du$$

Convolution Example



- Notice how the delta-function part at the right of the top "Function" produces an image of the 'Kernel'
- For a time series, that kernel defines the impulse response of the system
- For an imaging system, the kernel defines the point-spread function

Convolution Theroem

- The **convolution theorem** is extremely powerful and states that the Fourier transform of the convolution of two functions is the product of their individual Fourier transforms:

$$f * g \Leftrightarrow F \cdot G$$

- This theorem simplifies many problems: convolution is usually much more computationally intensive than multiplication, so by Fourier transforming first we can replace a complex operation by a simpler one

One and Two Dimensional Fourier Transforms

- The one-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \quad \text{where } j = \sqrt{-1}$$

Note the switch in notation from i, used earlier to j on this slide

  - Inverse Fourier transform:

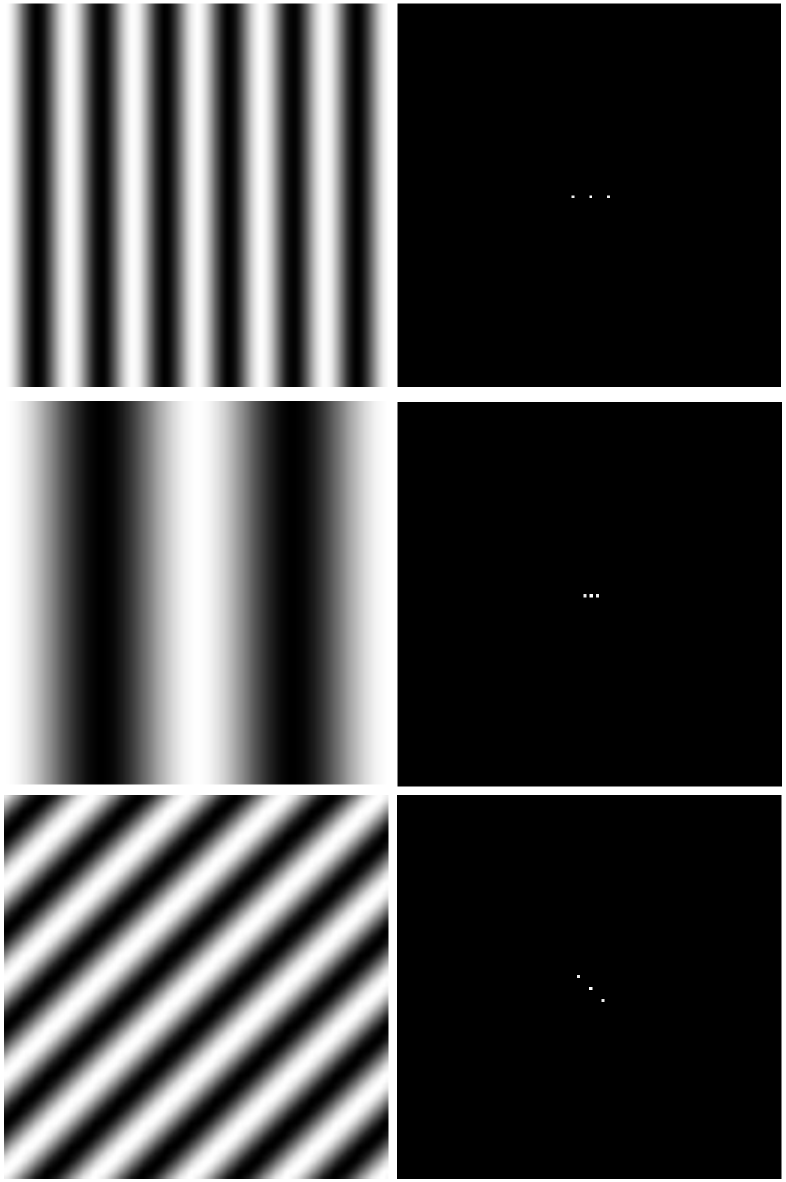
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)
  - Inverse Fourier transform:

$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

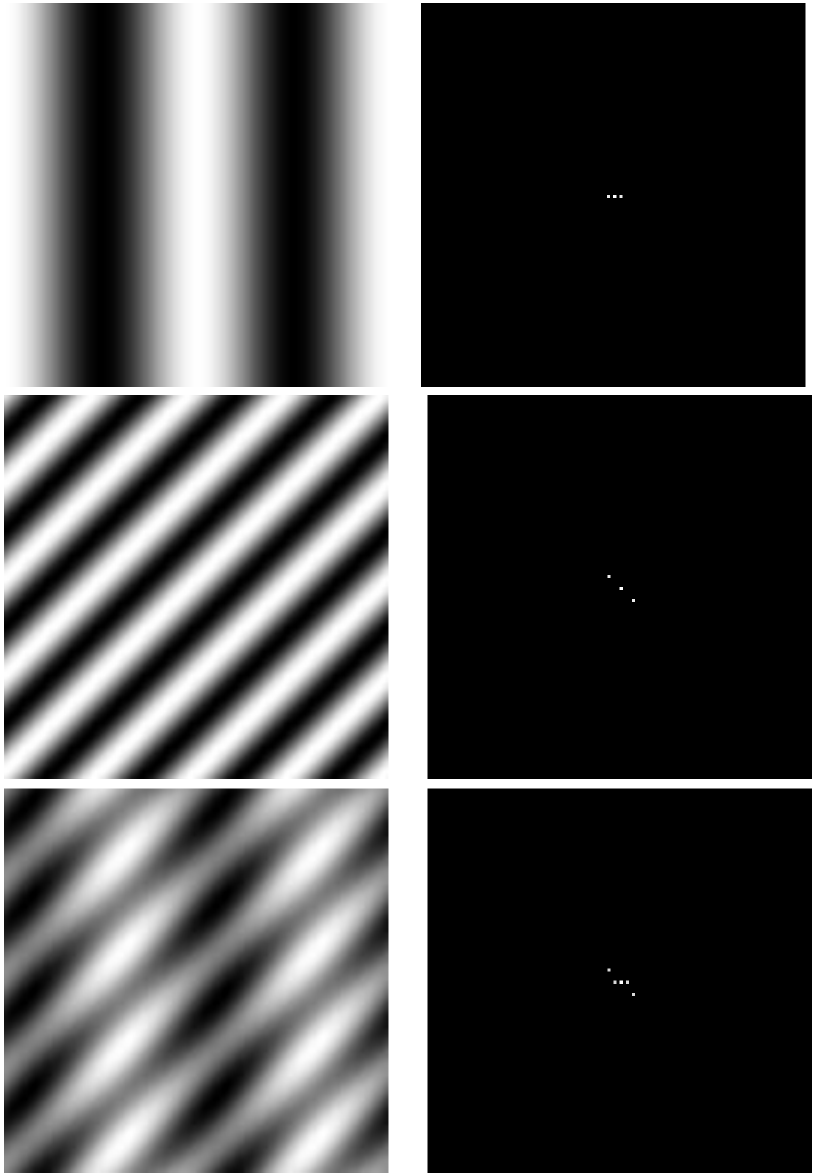
Fourier Transform of a 2-Dimensional Image



Images from Steven Lehar, Boston University

- Each image has three Fourier components
- The center pixel is the offset or DC term, while the other two encode the sinusoidal pattern
- The spacing of the sinusoidal pattern corresponds to the radius of the bright points from the center
- The direction of the sinusoidal pattern corresponds to the position angle of the bright points

Fourier Transform of a 2-Dimensional Image



- The bottom pair of images is the sum of the two above – the different Fourier components combine additively
- The brightness (left) and the Fourier images (right) are completely interchangeable – they contain exactly the same information.

Images from Steven Lehar, Boston University