



VieVS

Vienna VLBI and Satellite Software

Introduction to Least Squares Adjustment for geodetic VLBI

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Least Squares Adjustment why?

- observation is τ (baseline)
- unknown parameters are:
 - station positions
 - EOP
 - source positions
 - troposphere...

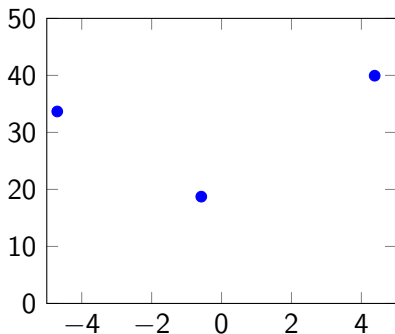
How do we link our observations to our unknown parameters?

A very simple example

- three sample points
- data fitting

- polynome of order 2:
 $y = ax^2 + bx + c$

- observations: (x_i, y_i)
- unknowns: a, b, c



A very simple example

$$(x_1, y_1) (x_2, y_2) (x_3, y_3)$$

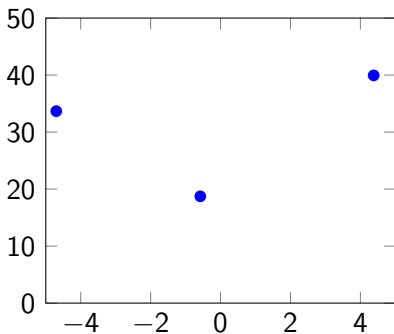
$$y_1 = ax_1^2 + bx_1 + c$$

$$y_2 = ax_2^2 + bx_2 + c$$

$$y_3 = ax_3^2 + bx_3 + c$$

Or in matrix notation:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$



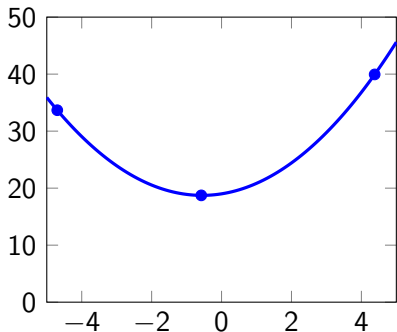
A very simple example

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

solve the equations

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

coefficients for
 $y = ax^2 + bx + c$



A very simple example

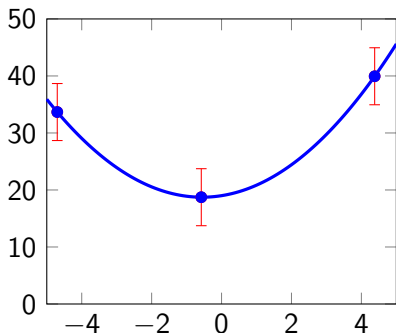
Warning!

Every observation has an error

How accurate are our coefficients a , b and c ?

What can we do?

If we have more observations we can average out the error



A very simple example

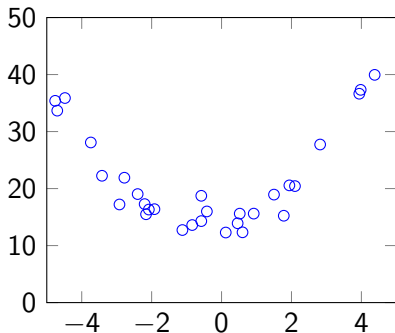
What happens if I have more observations?

$$y = ax^2 + bx + c$$

- three unknowns
 a, b, c (n_{unk})
- many observations
(n_{obs})

$$n_{obs} > n_{unk}$$

How do we get our unknown parameters?



A very simple example

simply pick three:

$$(x_1, y_1) \quad (x_2, y_2) \quad (x_3, y_3)$$

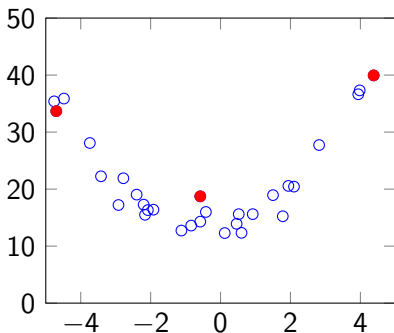
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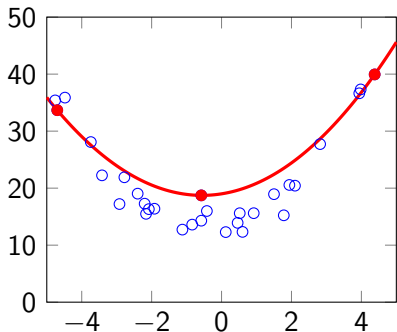
A very simple example

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

solve the equations

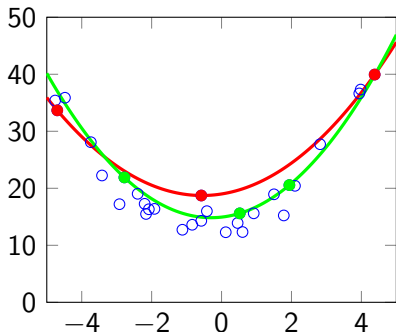
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

coefficients for
 $y = ax^2 + bx + c$



A very simple example

Why not this three?

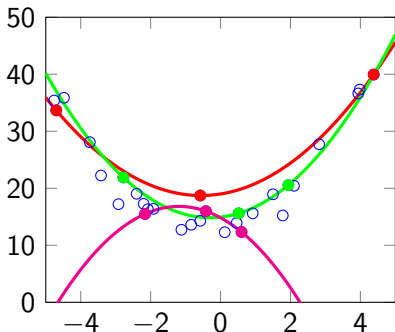


A very simple example

or this three:

Our goal:

we want to use every available observations



A very simple example

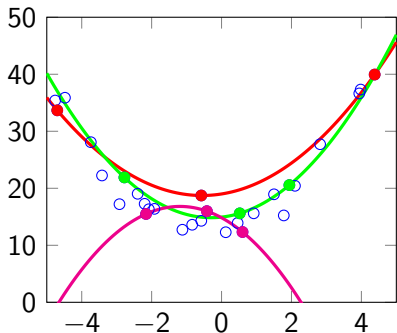
$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{\vec{I}} = \underbrace{\begin{pmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\vec{x}}$$

$$A \in \mathbb{R}^{n_{obs} \times n_{unk}}$$

$$\vec{x} \in \mathbb{R}^{n_{unk}}$$

$$\vec{I} \in \mathbb{R}^{n_{obs}}$$

$$A\vec{x} = \vec{I} \quad (1)$$



A very simple example

trick and solution

$$A^T A \vec{x} = A^T \vec{I} \quad (2)$$

$$\vec{x} = (A^T A)^{-1} A^T \vec{I} \quad (3)$$

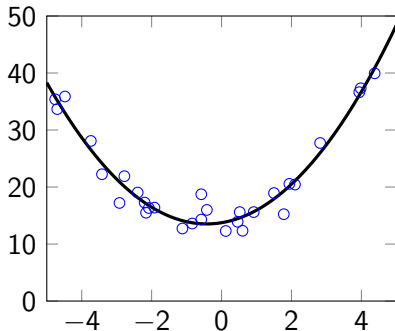
residuals v :

$$v_i = ax_i^2 + bx_i + c - y_i$$

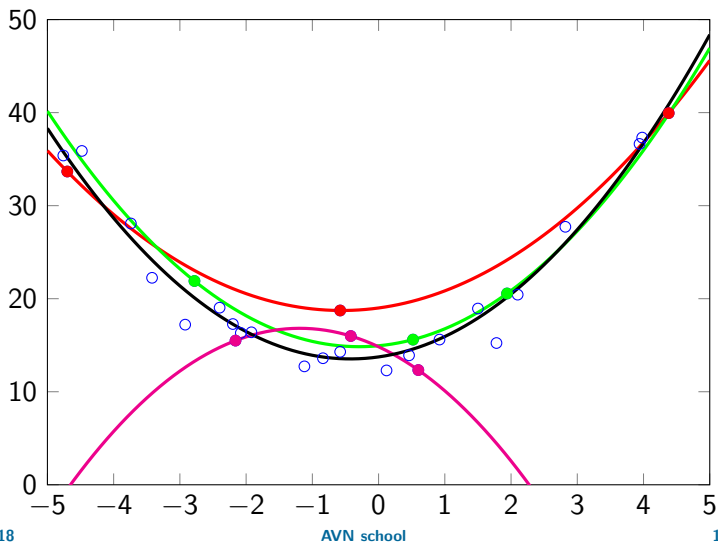
$$\vec{v} = A\vec{x} - \vec{I} \quad (4)$$

minimization

$$\sum v^2 = \vec{v}^T \vec{v} \quad (5)$$



A very simple example



A very simple example

conclusion / goal

- get the most reasonable values for our unknown parameters
- take all available information
- more observations than unknown parameters
- get some error margins

How can we do this?

Warning!

every observation has an error

Least Squares Methode

pros

- 😊 easy while powerful
- 😊 minimize residuals
- 😊 rigorous statistical approach
- 😊 individual weighting (28)
- 😊 accuracy information (32)
- 😊 manipulate geodetic datum (37)
- 😊 combine results of various LSMs (42)

cons

- 😞 only for linear functions
 - linearization via Taylor polynome (22)
 - a priori values
 - iterations (31)
- 😞 we need to model our observations $o - c$
- 😞 outliers

requirements for LSM

- partial derivatives (24)

What do we want to estimate?

- station coordinates
- source coordinates
- earth orientation parameters
- tropospheric parameters
- clock behavior
- and many more ...

Many unknown parameters...

What do we want to estimate?

Example: Scan with 5 stations $\rightarrow \frac{5 \cdot 4}{2} = 10$ observations

Estimates:

- Troposphere: (15p.)
 - zenith wet delay (5p.)
 - north gradient (5p.)
 - east gradient (5p.)
- clock (5p.)
- ...

Σ 20+ Parameters

Way more unknown parameters than observations ☹

Requirement for LSM

We need more observations than unknowns! $n_{obs} > n_{unk}$

What do we want to estimate?

Solution

piecewise linear offsets (34)

- we estimate parameters only every x minutes
 - linear interpolation in between
- reduces number of estimated parameters!

Improvement

constraints (35)

$$\blacksquare zwd_{sta1,03:00} = zwd_{sta1,04:00} \underbrace{\pm 1.5 \text{ [cm]}}_{\text{constraint}}$$

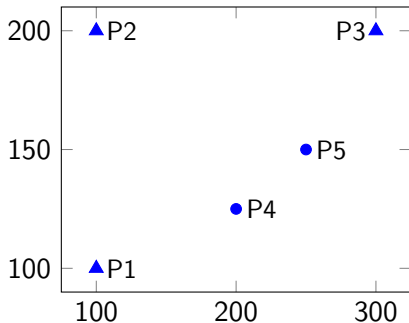
Conclusion

What should you remember?

- we calculate our solutions using Least Squares Adjustment
- we minimize (squared sum of) residuals
- we need to model our observations! → theoretical delay
 $o - c$
- we need partial derivatives for our Jacobian matrix A
- we need a priori values X_0 (usually not a big deal)
- we estimate additions x to a priori values X_0
- we need to eliminate outliers

Example 2

- 5 stations
- 3 with known coordinates
- 2 with unknown coordinates



Example 2

measured distance s_{ij}

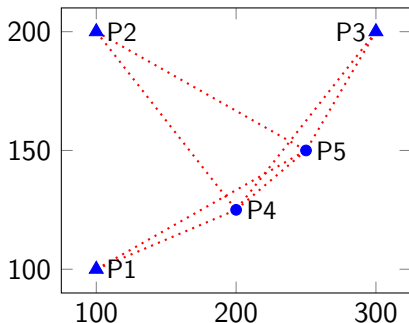
- unknowns:
 $(x_4, y_4) (x_5, y_5)$
 $\rightarrow n_{unk} = 4$
- observations: s_{ij}
 $\rightarrow n_{obs} = 7$

$$n_{obs} > n_{unk}$$

$$A \in \mathbb{R}^{7 \times 4}$$

$$\vec{x} \in \mathbb{R}^4$$

$$\vec{l} \in \mathbb{R}^7$$



Some basics about LSM linearization

We need to model our system:

$$s_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

not linear \rightarrow Taylor expansion:

$$\begin{aligned} T_{\infty, x_0}(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \end{aligned}$$

\rightarrow linearized observation equation

What do we need?

- We need partial derivative
- We need a priori values \vec{x}_0

Some basics about LSM definitions

- observations: L
- (computed) approximated observations: L_0
- reduced observation vector (observed minus computed): l

$$l = L - L_0 \quad (6)$$

- estimated unknowns: \hat{X}
- approximated unknowns: X_0
- reduced unknowns: x

$$x = \hat{X} - X_0 \quad (7)$$

- Designmatrix or Jacobian matrix: A (partials)

Back to Example 2

unknown: x_4, y_4, x_5, y_5

known: $x_1, y_1, x_2, y_2, x_3, y_3$

approximate values: $x_{4,0}, y_{4,0}, x_{5,0}, y_{5,0}$

$$s_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

$$A = \begin{pmatrix} \left(\frac{\partial s_{14}}{\partial x_4} \right) & \left(\frac{\partial s_{14}}{\partial y_4} \right) & \left(\frac{\partial s_{14}}{\partial x_5} \right) & \left(\frac{\partial s_{14}}{\partial y_5} \right) \\ \left(\frac{\partial s_{15}}{\partial x_4} \right) & \left(\frac{\partial s_{15}}{\partial y_4} \right) & \left(\frac{\partial s_{15}}{\partial x_5} \right) & \left(\frac{\partial s_{15}}{\partial y_5} \right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\partial s_{45}}{\partial x_4} \right) & \left(\frac{\partial s_{45}}{\partial y_4} \right) & \left(\frac{\partial s_{45}}{\partial x_5} \right) & \left(\frac{\partial s_{45}}{\partial y_5} \right) \end{pmatrix}$$

$$\frac{\partial s_{14}}{\partial x_4} = \frac{x_{4,0} - x_1}{\sqrt{(x_{4,0} - x_1)^2 + (y_{4,0} - y_1)^2}} \quad \frac{\partial s_{14}}{\partial y_4} = \frac{y_{4,0} - y_1}{\sqrt{(x_{4,0} - x_1)^2 + (y_{4,0} - y_1)^2}} \quad \frac{\partial s_{14}}{\partial x_5} = \frac{\partial s_{14}}{\partial y_5} = 0$$

Example 2

unknown: x_4, y_4, x_5, y_5

known: $x_1, y_1, x_2, y_2, x_3, y_3$

approximate values: $x_{4,0}, y_{4,0}, x_{5,0}, y_{5,0}$

reduced observation vector (observed minus computed)

$$l = L - L_0$$

$$l = \begin{pmatrix} s_{14} - s_{14,0} \\ s_{15} - s_{15,0} \\ \vdots \\ s_{45} - s_{45,0} \end{pmatrix}$$

$$s_{14,0} = \sqrt{(x_{4,0} - x_1)^2 + (y_{4,0} - y_1)^2}$$

Example 2

reduced observation vector (observed minus computed)

$$l = L - L_0$$

Jacobian matrix (partials): A

reduced unknowns: $x = X - X_0$

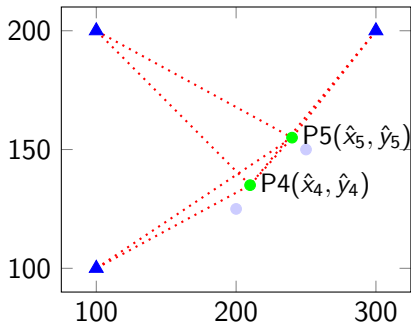
$$\begin{aligned} Ax = l &\xrightarrow{\text{trick}} A^T Ax = A^T l \rightarrow \\ x &= (A^T A)^{-1} A^T l \end{aligned} \quad (8)$$

What do we estimate?

We don't estimate whole unknown parameter \hat{X} but only additions x to some a priori values X_0

Example 2

The coordinates of our unknown points change



Some more basics about LSM covariance matrix

usually not every observations is equally accurate:

Covariance matrix $Q_{ll} \in \mathbb{R}^{n_{obs} \times n_{obs}}$, Weight matrix P :

$$Q_{ll} = \begin{pmatrix} \sigma_{l_1}^2 & \sigma_{l_1, l_2} & \dots & \sigma_{l_1, l_n} \\ \sigma_{l_1, l_2} & \sigma_{l_2}^2 & \dots & \sigma_{l_2, l_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{l_1, l_n} & \sigma_{l_2, l_n} & \dots & \sigma_{l_n}^2 \end{pmatrix} \quad (9)$$

if observations are independent $\sigma_{l_i, l_j} = 0$

$$Q_{ll} = \begin{pmatrix} \sigma_{l_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{l_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{l_n}^2 \end{pmatrix} \quad P = \underbrace{\frac{1}{\sigma_0^2}}_{\text{usually 1}} Q_{ll}^{-1}$$

Some more basics about LSM covariance matrix

Variance-Covariance matrix Q_{ll} , Weight matrix $P \in \mathbb{R}^{n_{obs} \times n_{obs}}$:

$$P = \underbrace{\frac{1}{s_0^2}}_{\text{usually 1}} Q_{ll}^{-1} = \begin{pmatrix} \frac{1}{\sigma_{l_1}^2} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_{l_2}^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sigma_{l_n}^2} \end{pmatrix} \quad (10)$$

good observation \rightarrow small standard deviation σ_{l_i} and variance $\sigma_{l_i}^2 \rightarrow$ high weight in P matrix

Some more basics about LSM equations

Final formulas:

$$\underbrace{A^T P A}_N x = \underbrace{A^T P l}_b \quad (11)$$

$$x = N^{-1} b \quad (12)$$

Important for LSM

- Normal equation matrix $N = (A^T P A)$ $N \in \mathbb{R}^{n_{unk} \times n_{unk}}$
- right hand side $b = A^T P l$ $b \in \mathbb{R}^{n_{unk}}$
- we minimize $v^T P v$

$$\hat{X} = X_0 + (A^T P A)^{-1} A^T P l = X_0 + N^{-1} b \quad (13)$$

Some more basics about LSM iteration

Sometimes a iteration is necessary.

- inaccurate a priori values X_0
- numerical reasons

You can use \hat{X} again as new a priori values X_0 and redo everything

$$X_{0,n+1} = \hat{X}_n = X_{0,n} + N_n^{-1} b_n \quad (14)$$

How good are our estimates?

covariance matrix

a posteriori variance factor σ_0^2 :

$$\sigma_0^2 = \frac{v^T P v}{n_{obs} - n_{unk}} \quad (15)$$

remember our residuals v are: $v = Ax - l$

Variance-Covariance Matrix $Q_{xx} \in \mathbb{R}^{n_{unk} \times n_{unk}}$

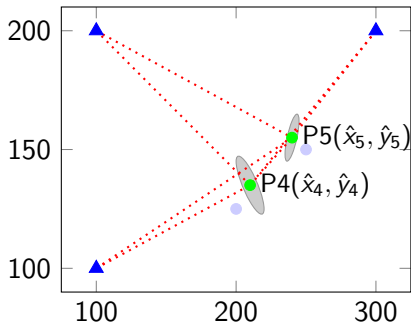
$$Q_{xx} = \sigma_0^2 N^{-1} = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1, x_2} & \cdots & \sigma_{x_1, x_n} \\ \sigma_{x_1, x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2, x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_1, x_n} & \sigma_{x_2, x_n} & \cdots & \sigma_{x_n}^2 \end{pmatrix} \quad (16)$$

$\sigma_{x_i}^2$ is variance of unknown parameter x_i

σ_{x_i} is standard deviation

Back to example 2

Now we know how accurate our coordinates are



Concepts What are piecewise linear offsets?

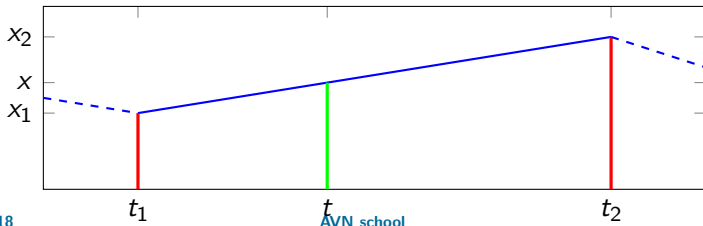
time dependent parameters

Example: Troposphere (mf = mapping function)

$$\Delta L_w(t) = mf_w(t)x_1 + mf_w(t) \frac{t - t_1}{t_2 - t_1} (x_2 - x_1)$$

$$\frac{\partial \Delta L_w}{\partial x_1} = mf_w(t) - mf_w(t) \frac{t - t_1}{t_2 - t_1}$$

$$\frac{\partial \Delta L_w}{\partial x_2} = mf_w(t) \frac{t - t_1}{t_2 - t_1}$$



Concepts What are constraints?

- pseudo observations
- help us with singularities
- useful concept

A simple example: (think of troposphere between two time epochs)

$$\begin{aligned}x_j - x_i &= 0 \pm 5 \\x_{j-1} - x_{i-1} &= 0 \pm 3\end{aligned}$$

(\pm means standard deviation)

$$A_c \in \mathbb{R}^{n_{const} \times n_{unk}}$$

$$A_c = \begin{pmatrix} \dots & 0 & -1 & 0 & \dots & 0 & 1 & 0 & \dots \\ \dots & -1 & 0 & 0 & \dots & 1 & 0 & 0 & \dots \end{pmatrix}$$

$$l_c \in \mathbb{R}^{n_{const}}$$

$$P_c \in \mathbb{R}^{n_{const} \times n_{const}}$$

$$l_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$P_c = \begin{pmatrix} \frac{1}{5^2} & 0 \\ 0 & \frac{1}{3^2} \end{pmatrix}$$

Concepts What are constraints?

$$N_c \in \mathbb{R}^{n_{unk} \times n_{unk}}$$

$$N_c = A_c^\top P_c A_c \quad (17)$$

$$b_c \in \mathbb{R}^{n_{const}}$$

$$b_c = A_c^\top P_c l_c \quad (18)$$

Combination with the real observations:

$$N_{total} = A^\top P A + A_c^\top P_c A_c = N + N_c \quad (19)$$

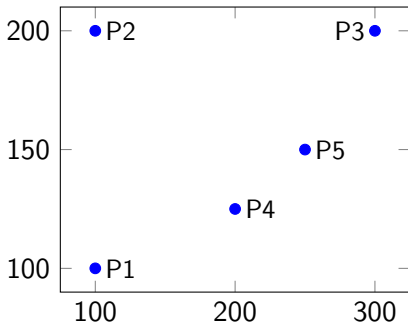
$$b_{total} = A^\top P l + A_c^\top P_c l_c = b + b_c \quad (20)$$

The only thing that changes is:

$$\sigma_{0,const}^2 = \frac{v^\top P v + v_c^\top P_c v_c}{n_{obs} + n_{const} - n_{unk}} \quad (21)$$

Concepts What is a geodetic datum? - Example 2

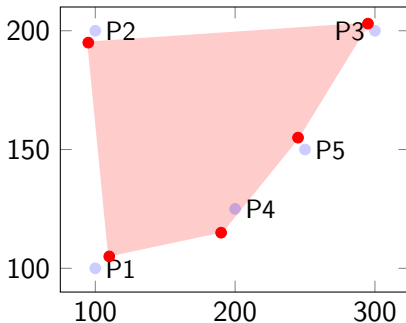
- Think of example 2
- This time no station has fixed coordinates
- unknowns: all coordinates
- observations: s_{ij}
- we have a priori coordinates



Concepts What is a geodetic datum?

- we know inner geometry very well
- we can not estimate absolute coordinates
- N matrix is singular

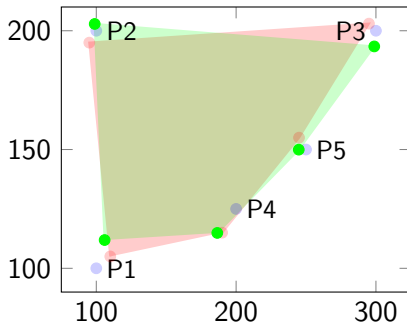
is this correct?



Concepts What is a geodetic datum? - Example 2

- we know inner geometry very well
- we can not estimate absolute coordinates
- N matrix is singular

or this?



Concepts What is a geodetic datum? - Formulas

We need datum stations with additional conditions for example in this case we could say:

$$\begin{array}{ll}
 \text{translation in } x & \sum dx_i = 0 \\
 \text{translation in } y & \sum dy_i = 0 \\
 \text{rotation around } z & \sum (y_i dx_i - x_i dy_i) = 0 \\
 \text{scale} & \sum (x_i dx_i - y_i dy_i) = 0
 \end{array}$$

This would lead to the following matrix: $G \in \mathbb{R}^{n_{datum} \times n_{unk}}$

$$G = \begin{pmatrix} 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & \dots & 0 & 1 \\ y_1 & -x_1 & \dots & y_n & -x_n \\ x_1 & y_1 & \dots & x_n & y_n \end{pmatrix} \quad (22)$$

Concepts What is a geodetic datum? - Solution

The solution is now:

$$\begin{pmatrix} A^T P A & G \\ G^T & 0 \end{pmatrix} \begin{pmatrix} x \\ k \end{pmatrix} = \begin{pmatrix} A^T P l \\ 0 \end{pmatrix} \quad (23)$$

What else is different:

$$\sigma_0^2 = \frac{v^T P v}{n_{obs} + n_{datum} - n_{unk}}; \quad \begin{pmatrix} Q_{xx} & Q_{xk} \\ Q_{xk} & Q_{kk} \end{pmatrix} = \sigma_0^2 \begin{pmatrix} A^T P A & G \\ G^T & 0 \end{pmatrix}^{-1} \quad (24)$$

Geodetic datum in VLBI

usually we use NNT/NNR - No Net Rotation and No Net Translation conditions (3 NNT and 3 NNR) scale is fixed by observations

Concepts What is a global solution?

- combination of large number of sessions to estimate accurate common parameters
- combination is done at the normal equation level
- you need to make sure N matrix has same structure
 → you want to get rid of some parameters

$$N_x = b \quad \rightarrow \quad \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\underbrace{\left(N_{11} - N_{12} N_{22}^{-1} N_{21} \right)}_{N_{reduc}} x_1 = \underbrace{b_1 - N_{12} N_{22}^{-1} b_2}_{b_{reduc}} \quad (25)$$

$$N_{REDUC} = N_{reduc_1} + N_{reduc_2} + \dots + N_{reduc_n} \quad (26)$$

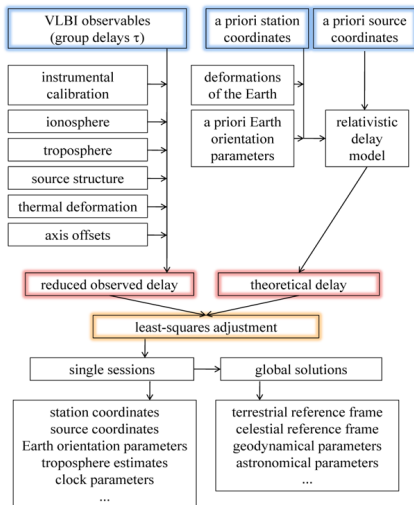
$$b_{REDUC} = b_{reduc_1} + b_{reduc_2} + \dots + b_{reduc_n} \quad (27)$$

Conclusion

What is to remember?

- we calculate our solutions using Least Squares Adjustment
- we minimize $v^T P v$
- we need to model our observations! → theoretical delay
- we need partial derivatives for our Jacobian matrix A
- we need a priori values X_0 (usually not a big deal)
- we estimate additions x to a priori values X_0

Link to VLBI





VieVS

Vienna VLBI and Satellite Software

Lecture LSM

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