VLBI System Documentation
Mark IV Field System

Pointing Model Derivation

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Operations Manual

NASA/Goddard Space Flight Center
Space Geodesy Project

Version 8.2
September 1, 1993

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1.0 Introduction

Pointing models are used to correct the pointing of antennas and telescopes for deviations from ideal construction. The model accounts for a variety of mechanical and electrical errors.

This document describes the standard pointing model used in the Field System. This model is similar in form to pointing models used by many radio telescopes. This particular model, in terms of the choices of signs and which terms are estimated is a variant of the model used to analyze Apollo 13 tracking data (*Apollo 13 MSFN Metric Tracking Performance, Final Report,* Document X-832-70-156, NASA Goddard Space Flight Center, Greenbelt, MD 20771, July 1970). A brief description of the model is given in the **Pointing Model File** manual.

There are three sections in this manual. The first states the model with an explanation of each term including a physical interpretation. The second section discusses some of the details of using the model. The third section explains, in some detail, the steps used to derive the model.

2.0 Pointing Model

The current model contains 14 terms. Every term is not used by every telescope. The model gives values for "observed-minus-calculated" angles. In other words, the model calculates an offset which is added to the calculated command angles to get the corrected command angles, which actually will point the antenna in the desired direction.

The nomenclature will be that the corrected angles are the "apparent" angles, and the uncorrected angles, which refer to a fixed local reference frame will be the "true" angles. Consequently, another way of saying "observed-minus-calculated" is "apparent-minus-true".

One of the differences between the model used here and for the Apollo 13 data (see reference in section 1.0 of this document) is that the model has been generalized to include all the antenna axis types used by the Crustal Dynamics Project: XY (both NS and EW), AZEL, and HADC. The basic similarity between these coordinate systems is that the first coordinate is always a longitude-like coordinate and the second coordinate is always a latitude-like coordinate. The difference is that the XY angle systems are right-handed and the AZEL and HADC angle systems are left-handed if an outward radial direction is assumed for the third coordinate. Otherwise all the antenna coordinates represent the same system rotated to different orientations. The Goddard X-Document: *Mathematical Relationships of the MFOD Antenna Axes* (Document X-553-67-123, NASA Goddard Space Flight Center, Greeenbelt, MD 20771, May 1967) gives the coordinate relationships for the different axis types as well as a general procedure for determining transformations from one system to another. The **Coordinate Conversions** manual collects the results of this X-Document for easy reference.

The longitude-like coordinate (X, AZ, HA) correction ΔX, expressed for X and Y angles is:

$$\begin{aligned}\Delta X = P_1 &- P_2 \cos\phi \sin X \sec Y + P_3 \tan Y - P_4 \sec Y + P_5 \sin X \tan Y \\ &- P_6 \cos X \tan Y + P_{12} X + P_{13} \cos X + P_{14} \sin X \\ &+ P_{15} \cos 2X + P_{16} \sin 2X\end{aligned} \tag{1}$$

The latitude-like coordinate (Y, EL, DC) correction ΔY, expressed for X and Y angles is: where:

$$\Delta Y = P_5\cos X + P_6\sin X + P_7 - P_8(\cos\phi\cos X\sin Y - \sin\phi\cos Y) + P_9 Y + P_{10}\cos Y + P_{11}\sin Y \quad (2)$$

ϕ is the angle between the Y=+90° and the horizon, measured positive in the direction from Y=+90° to X,Y=(0°,0°). For a nominal XY (NS or EW) system, ϕ=0.0°; for a nominal AZEL, ϕ=90.0°; for a nominal HADC, ϕ=geodetic latitude.

P_1 is the X-angle offset, the difference of the X-angle encoder bias (positive if encoder reading is too high) minus 'tilt around', which is the tilt of the antenna around the Y=+90° (positive if apparent X,Y=(0°,0°) is closer to true X,Y=(+90°,0°))

P_2 is the X angle sag, the effect of gravity on the RF axis of the dish projected on the X direction (positive if the RF axis is lower)

P_3 is the perpendicular axis skew, the apparent Y=+90° to true Y=0° plane lack of orthogonality in the plane perpendicular to the current X angle meridian (positive if apparent Y=+90° is closer to true X,Y=((current X)-90°,0°))

P_4 is the box offset, RF-axis to radial direction misalignment along the X direction (positive if RF-axis is toward the increasing X angle direction)

P_5 is the "tilt out", tilt of the apparent Y=+90° toward the true X,Y=(0°,0°) position (positive if apparent Y=+90° is closer to true X,Y=(0°,0°))

P_6 is the "tilt over", tilt of the Y=+90° toward the X,Y=(+90°,0°) position (positive if apparent Y=+90° is closer to true X,Y=(+90°,0°))

P_7 is the Y angle offset, difference of the Y angle encoder bias (positive if encoder reads too high) minus the sum of the skew of Y=+90° along the current X meridian angle (positive if apparent Y=+90° is farther from true X,Y=(current X,0°)) plus the RF axis to radial direction misalignment along the Y direction (positive if the RF axis is toward the increasing Y angle direction)

P_8 is the Y angle sag, the effect of gravity on the RF axis of the dish projected on the Y direction (positive if the RF axis is lower)

P_9 is an *ad hoc* Y-angle excess scale factor (greater than 0 if the encoder read-out changes faster than the actual antenna position)

P_{10} is an *ad hoc* $\Delta Y \cos Y$ coefficient

P_{11} is an *ad hoc* $\Delta Y \sin Y$ coefficient

P_{12} is an *ad hoc* X-angle excess scale factor (greater than 0 if the encoder read-out changes faster than the actual antenna position)

P_{13} is an *ad hoc* $\Delta X \cos X$ coefficient

P_{14} is an *ad hoc* $\Delta X \sin X$ coefficient

P_{15} is an *ad hoc* $\Delta X \cos 2X$ coefficient

P_{16} is an *ad hoc* $\Delta X \sin 2X$ coefficient

3.0 Discussion

The terms of the model break down into two classes: (1) those that model some misalignment in the coordinate system, and (2) gravitational deformation and other, perhaps *ad hoc*, terms. Among these two classes: P_1, P_3, P_4, P_5, P_6, and P_7 represent misalignments of the antenna and feed system from ideal, all the other terms represent gravitational deformation and/or *ad hoc* terms. P_2 and P_8 represent the effect of a gravitational sag of the antenna. However, depending on the actual structure of the antenna other *ad hoc* terms may be necessary and the gravitational effects may also contaminate other terms. This is discussed in more detail below.

The interpretations of model terms in the previous section were written somewhat obliquely, so that their interpretation is independent of axis type. X could be globally replaced with AZ or HA and Y replaced with EL or DC respectively with no change in the interpretation of the terms. The terms of the model form a fairly comprehensive set of the possible misalignments of the antenna: tilts of the axis system, RF axis misalignments, skewing between the two coordinate axes, and biases in the encoder read-out.

Terms P_5 and P_6, or "tilt out" and "tilt over", represent two orthogonal components of the tilt of a coordinate system's equatorial plane. Table 3.1 contains a summary of the interpretation of the two terms in local coordinates for different axis systems. For an XYNS, tilt out is the displacement of the apparent $Y=+90°$ *up* from true $Y=+90°$ and the tilt over is its displacement *east*. Similarly for an AZEL system, tilt out is the displacement of apparent $EL=+90°$ to the north and tilt over is its displacement to the east. For a HADC system, not at the north or south pole, tilt out is the displacement of apparent $DC=+90°$ toward zenith, tilt out is its displacement toward +6 hours local hour angle. Finally, for an XYEW system, tilt out is the displacement of apparent $Y=+90°$ *up*, tilt over is its displacement to the south. The use of up, east, zenith, six hours local hour angle, and south may not be accurate if ϕ does not have the nominal value for its coordinate system, or for a nominal HADC antenna at the north or south pole.

Table 3.1 Tilt Directions Displacment of apparent *Y* (or *EL* or *DC*) = +90° corresponding to a positive parameter value.					
Term	Name	XYNS (ϕ=0°)	XYEW (ϕ=0°)	AZEL (ϕ=90°)	HADC (-90° $< \phi <$ +90°)
P_5	tilt out	up	up	north	zenith
P_6	tilt over	east	south	east	+6 hours HA

The P_2 and P_8 terms represent the same geometrical effect, gravitational deflection, however they are estimated independently. The actual deflection may be different in different directions depending on the symmetry and rigidity of the dish. Hence the deflection along the X-direction may differ from that along the Y-direction. An effective argument could be made that these terms should be estimated as one and *ad hoc* terms used to account for any differences.

In general there is no *a priori* model for gravitational deflection. For a perfectly symmetric dish and backup structure the terms P_2 and P_8 should be the same and completely model the sag. However, since no dish is perfect, these terms will differ and may in fact be inadequate for modeling the deformations. Depending on the details of the dish, the true deflection terms may have a form similar to other terms already being estimated. In terms of increasing complexity the true, but unknown, deformation terms are: (1) identically 0, perfectly rigid dish; (2) P_2 and P_8 not zero but identical, dish and structure fairly rigid and deform the same way along both coordinate axes; (3) P_2 and P_8 different and other terms are not required, dish and structure fairly rigid and deform differently along the two axes; (4) P_2 and P_8 different and other, *ad hoc*, terms anti-symmetric about 0° zenith angle are necessary, dish less rigid but behaves repeatably; (5) P_2 and P_8 different and other, *ad hoc*, terms both symmetric and anti-symmetric about 0° zenith angle are necessary, dish less rigid and less symmetric but behaves repeatably; and (6) pointing not repeatable, complicated hysteresis. Although gravitational deformation may lead to a need to estimate additional terms, *ad hoc* terms may be necessary for other reasons. Conversely, depending on the form of the deformation, it may change the apparent value of other terms. Because of these problems, determinig a model for gravitational deformation may, in absence of a good structural model for an antenna, turn into an *ad hoc* procedure of simply trying to remove any systematic effects that are left in the residuals. Whether or not this is acceptable depends on the application the pointing model will be used for. If the only requirement is to achieve accurate pointing then such an *ad hoc* strategy may be the best choice anyway.

The terms that represent excess scale factors in X and Y, P_{12} and P_9 are not likely to be needed for encoder and synchro read antenna positions since there should be no scale factor error is such systems. They may however be useful as *ad hoc* parameters since they each correspond to a slope.

When pointing in plunge, *i.e.* Y-angles greater than +90°, some care must be exercised in using this model. To establish definitions: "pointing in plunge" or "plunging the antenna" refers to the Y coordinate exceeding +90°. The term "plunged coordinates" refers to X and Y angles, where the Y angle would exceed +90°, but have been modified so that the Y angle is less than +90°; "natural coordinates" will refer to raw or unfixed angles in which Y could, and may or may not, exceed +90°. For example a natural coordinate position of X,Y=(+45°,+135°) would be represented by plunged coordinates X,Y=(-135°,+45°). When the antenna is plunged, positive Y offsets will increase the natural Y coordinate or equivalently decrease the plunged Y coordinate. For plunged coordinates, some adjustment of the signs of the model terms may be necessary. The corrections can be determined by considering what the effective correction would be. The simplest thing to do is to stick with natural coordinates. For natural coordinates, all the terms in the model have the correct sign regardless of whether the antenna is plunged.

Other modifications of the reported coordinates are possible. The antenna may "reverse plunge" or go to Y angles less than -90°. Also there may be more than one full rotation or "wrap" in the X direction. The recommendation for these cases is to continue to use the natural antenna coordinates, reporting Y angles below -90° and X angles with the addition or subtraction of multiples of 360° to account for the wraps. There may be antennas for which certain *ad hoc* model terms would be more naturally represented in "plunged" rather the "natural" coordinates. For those cases, however, it is recommended that the *ad hoc* terms be modified to take this into account and that natural antenna coordinates be used for the raw data.

4.0 Model Derivation

This section summarizes the mathematical steps needed to formulate the pointing model. The first eight terms of the model represent simple geometric effects that can be derived with a simple application of vector analysis. For a given coordinate system, say XYNS, the usual strategy is: (1) take an arbitrary true XYNS pointing direction and convert it to a direction cosines representation in a cartesian coordinate system aligned with the pole and equator of the XYNS system, (2) manipulate the cartesian coordinates with rotations to mimic the desired physical effect, (3) gather terms and simplify using the small angle approximation and discarding terms of quadratic or higher order in small angles. This derivation points out the limitations of the model: the parameters are required to have small values so that they may be considered independent of rotations and may not be used too close to the poles of the coordinate system.

There are two problems with the model near the poles: (1) The correction may require the Y angle to plunge over the top, if the antenna cannot plunge, then the point may be effectively unreachable. (2) The closer the Y value is to the pole, the smaller the range of X angles over which the assumption that X and Y angles are perpendicular is a good approximation. The later problem reflects the small angle approximation and linear approximations breaking down. Terms P_2 through P_6 and P_8 fail in this way. This is not too serious an obstacle since most antennas have difficulty tracking, if they can point at all, near the poles.

If corrections are required near the poles for terms P_2 through P_6 and P_8 they can be calculated directly by applying the appropriate rotations without any approximations. Of course for corrected Y angles that are at the pole, the X angle is indeterminate. The derivations for all terms except P_2 and P_8 start by constructing the exact correction and then linearizing. The exact expression can be solved for the corrected angle. In this section, $=$ and $\approx$ are used carefully for exact and approximate equality.

In some instances derivation of the linearized pointing corrections requires division by $\cos Y$ or $\sin X$ or $\cos X$. Division by $\cos Y$ is unavoidable and makes such terms invalid at the poles, which is not a great problem since most of them are not valid even in the region near the pole. Division by $\cos X$ or $\sin X$ makes it impossible to verify the terms along the meridians of $\pm 90°$ or $0°$ and $180°$ respectively. However, for these cases it is always possible to verify the expression for the problem meridians by considering an alternate derivation.

4.1 Definitions and Identities

The cartesian coordinate system aligned with the XYNS axis system will be (ijk). $+i$ will be aligned along $X,Y=(0°,0°)$ true. $+j$ will be aligned along $X,Y=(+90°,0°)$ true. $+k$ will be along $Y=+90°$ true. Apparent coordinates will be represented by priming the corresponding designator. Converting from XYNS to (ijk) or vice-versa is accomplished by:

$$\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} \cos Y \cos X \\ \cos Y \sin X \\ \sin Y \end{bmatrix} \qquad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \tan^{-1}\frac{j}{i} \\ \sin^{-1}k \end{bmatrix} \tag{3}$$

The $\tan^{-1}$ in this case must retain the correct quadrant. A straightforward way to do this is to use the half-angle identity. The rotation matrices for a counterclockwise, considered positive, rotation of the coordinate axes about the $+i$, $+j$, or $+k$ axes by an angle θ are:

$$R_i(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad R_j(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_k(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

In the derivations below, the effects of pointing errors are simulated by rotating the coordinate axes to the place where they would "appear-to-be."

Use is made of the following trigonometric identities:

$$\sin(-A) = -\sin A \tag{5}$$

$$\cos(-A) = \cos A \tag{6}$$

$$\cos^2 A + \sin^2 A = 1 \tag{7}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{8}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad (9)$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \qquad (10)$$

The small angle approximation is that for sufficiently small angles A:

$$\sin A \approx A \qquad \textit{and} \qquad \cos A \approx 1 \qquad (11)$$

Discarding terms of quadratic or greater order in B for small values of B is an assumption that these terms can be safely neglected because:

$$B \gg B^2 \qquad (12)$$

By definition:

$$\Delta Y = Y' - Y \qquad \textit{and} \qquad \Delta X = X' - X \qquad (13)$$

4.2 Encoder Biases

These are the simplest terms since they are constants. In some sense they are instrumentation errors rather than alignment errors. Suppose that the encoder on a given axis always reads high by a fixed amount. Then clearly the angles reported by that encoder should be high by that same fixed amount:

$$\Delta X = P_1 \qquad \textit{and} \qquad \Delta Y = P_7 \qquad (14)$$

4.3 Coordinate Axis Tilts

As long as we consider only a small tilt we can model an arbitrary tilt as three independent rotations: (1) P_6 tilt over, a rotation around $-i$, (2) P_5 tilt out, a rotation around $+j$, and (3) P_1 tilt around, a rotation around $+k$.

Term P_6, tilt over, is a positive rotation of the axis system around $-i$ or a negative rotation of the coordinate system around $+i$. A rotation matrix that will transform the true (ijk) coordinates to apparent coordinates in the $(ijk)'$ system, using equation (4) is:

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos P_6 & -\sin P_6 \\ 0 & \sin P_6 & \cos P_6 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \qquad \textbf{(15)}$$

Examining coordinate k' and using equation (3), we get an expression for Y':

$$\sin Y' = \sin P_6 \cos Y \sin X + \cos P_6 \sin Y \qquad \textbf{(16)}$$

Reducing this to a linear form is a multi-step process. Since a similar reduction is necessary for most of the terms, it would be very tedious to do this in complete detail for all of the terms. However, as an example, the steps in linearizing equation (16) are: use equation (11) for P_6, use equation (10), use $Y'+Y=2Y+\Delta Y$, use equation (8), use equation (11) for $\Delta Y/2$, then use equation (12) for ΔY, and finally divide by $\cos Y$. The result is:

$$\Delta Y \approx P_6 \sin X \qquad \textbf{(17)}$$

Examining coordinate j' and using equation (3), we get an expression for X':

$$\cos Y' \sin X' = \cos P_6 \cos Y \sin X - \sin P_6 \sin Y \qquad \textbf{(18)}$$

Approximating and using the already derived expression in equation (17) for ΔY leads to the linearized expression:

$$\Delta X \approx -P_6 \tan Y \cos X \qquad \textbf{(19)}$$

Derivation of the linear ΔX term requires division by $\cos X$, making the result impossible to verify for $X = \pm 90°$. However, if i' had been considered, division by $\sin X$ would be required instead, which would have been impossible to verify for $X=0°$ and $X=\pm 180°$. In combination these two results show that the expression is valid for all X.

Term P_5, tilt out, is a positive rotation of the axis system around $+j$. From equation (4), a rotation matrix that will transform the true (ijk) coordinates to apparent coordinates in the $(ijk)'$ system is:

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos P_5 & 0 & -\sin P_5 \\ 0 & 1 & 0 \\ \sin P_5 & 0 & \cos P_5 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{20}$$

Examining k' and using equation (3) we get an expression for Y':

$$\sin Y' = \cos P_5 \sin Y + \sin P_5 \cos Y \cos X \tag{21}$$

which approximates to the linear form:

$$\Delta Y = P_5 \cos X \tag{22}$$

Examining i' and using equation (3) gives the complete expression for X':

$$\cos Y' \cos X' = \cos P_5 \cos Y \cos X - \sin P_5 \sin Y \tag{23}$$

Approximating and using the already derived expression for ΔY gives the linearized form:

$$\Delta X \approx P_5 \tan Y \sin X \tag{24}$$

Division by $\sin X$ in deriving the ΔX term here can be overcome in an analogous fashion to overcoming division by $\cos X$ in the derivation of P_6.

Term P_1, tilt around, is a positive rotation of the axis system around $+k$, a rotation matrix that will transform the true (ijk) coordinates to apparent coordinates in the $(ijk)'$ system, is:

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos P_1 & \sin P_1 & 0 \\ -\sin P_1 & \cos P_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{25}$$

Examining k', we get:

$$\sin Y' = \sin Y \tag{26}$$

which implies, without approximation, that:

$$\Delta Y = 0 \tag{27}$$

Examining j', we get:

$$\cos Y' \sin X' = \cos P_1 \cos Y \sin X - \sin P_1 \cos Y \cos X \tag{28}$$

Using the already derived fact that $Y' = Y$ gives, without approximation:

$$\Delta X = -P_1 \tag{29}$$

4.4 Axis Skew

Axis skew represents a misalignment between the two axes. The skewing could be thought of as tilt of $Y = +90°$ relative to the $Y = 0°$ plane. However it differs from the tilts derived previously in that the phase of the X angle with respect to the tilt direction always appears to be the same. We can view the skewing as having two orthogonal components, one along a meridan of X=90° from the current X angle, the other along the current X meridian. The effect for a point along the $X = 0°$ meridian corresponds to the effect of tilt over and tilt out, respectively.

For a skewing P_3 of $Y = +90°$ toward $X,Y = (-90°, 0°)$, opposite sense of tilt over, equations (16) and (18), with $X = 0°$, give:

$$\sin Y' = \cos P_3 \sin Y \quad \textit{and} \quad \cos Y' \sin \Delta X = \sin P_3 \sin Y \tag{30}$$

The linearized forms are:

$$\Delta Y \approx 0 \quad \textit{and} \quad \Delta X \approx P_3 \tan Y \tag{31}$$

Equations (30) and (31) are independent of the actual value of the X angle as expected. They represent a skewing of $Y = +90°$ along a meridian of X=90° from the current meridian.

The other component of skew, by a similar argument, for a point along $X=0°$ is modeled by a tilt of $Y=+90°$ along $X=0°$ meridan. For a skew of P_7 of $Y=+90°$ away from $X,Y=(0°,0°)$, opposite sense of tilt out, equations (21) and (23), with $X=0°$, give:

$$\sin Y' = \cos P_7 \sin Y - \sin P_7 \cos Y \quad \textit{and} \quad \cos Y' \cos X' = \cos P_7 \cos Y + \sin P_7 \sin Y \tag{32}$$

Reduced forms are obtained without approximations:

$$\Delta Y = -P_7 \qquad \textit{and} \qquad \Delta X = 0 \tag{33}$$

Again the particular value of the X angle does not matter. This represents a skewing of $Y=+90°$ along the current meridian of X. The effect is the same, except for the sign, to a Y angle encoder offset.

4.5 Box Offsets

The box offsets represent the angular misalignment of the RF axis relative to the current X-angle meridian and Y-angle parallel. Consider a RF misalignment perpendicular to the current X-angle meridian, This effect would be independent of the particular X-angle meridian but would depend on the Y angle since meridians are "bunched together" near the pole. Without loss of generality we will consider a point on the $X=0°$ meridian. To construct the appropriate transformation, three steps are required: (1) Rotate around $+j$ by an angle -Y to represent the point in a reference frame where it is on the equator; (2) rotate by an an angle P_4 around the new $+k$ axis to represent the effect of the beam being P_4 too far in the +X direction; (3) perform the inverse of rotation (1) to place the point back in the original reference frame:

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos Y & 0 & -\sin Y \\ 0 & 1 & 0 \\ \sin Y & 0 & \cos Y \end{bmatrix} \begin{bmatrix} \cos P_4 & \sin P_4 & 0 \\ -\sin P_4 & \cos P_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos Y & 0 & \sin Y \\ 0 & 1 & 0 \\ -\sin Y & 0 & \cos Y \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos P_4 \cos^2 Y + \sin^2 Y & \sin P_4 \cos Y & \cos Y \sin Y (\cos P_4 - 1) \\ -\sin P_4 \cos Y & \cos P_4 & -\sin P_4 \sin Y \\ \cos Y \sin Y (\cos P_4 - 1) & \sin P_4 \sin Y & \cos P_4 \sin^2 Y + \cos^2 Y \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{35}$$

Assuming $X=0°$, we get an exact expression for Y',by examining k':

$$\sin Y' = \sin Y \cos P_4 \tag{36}$$

or if we approximate, we get the linear expression:

$$\Delta Y \approx 0 \tag{37}$$

If we examine j', again assuming $X=0°$, we get an exact expression for X':

$$\cos Y' \sin \Delta X = -\sin P_4 \tag{38}$$

If we approximate and use the linearized result $\Delta Y=0°$, we get:

$$\Delta X \approx -P_4 \sec Y \tag{39}$$

Considering a RF misalignment along the current X meridian, there is no loss of generality in considering a point on the $X=0°$ meridian. If we rotate that point by an amount $-P_7$ around +j to simulate the RF axis being too far toward $Y=+90°$ then:

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos P_7 & 0 & \sin P_7 \\ 0 & 1 & 0 \\ -\sin P_7 & 0 & \cos P_7 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{40}$$

If we examine k', by assumption $X=0°$, we get:

$$\sin Y' = \cos P_7 \sin Y - \sin P_7 \cos Y \tag{41}$$

which implies, without approximation:

$$\Delta Y = -P_7 \tag{42}$$

Examining j' and using the assumption that $X=0°$ gives:

$$\cos Y' \sin X' = 0 \tag{43}$$

which implies without approximation:

$$\Delta X = 0 \tag{44}$$

4.6 Gravitational Sag Terms

Terms P_2 and P_8 which are elevation angle deflections, have a slightly more involved derivation than the previous terms. Consider an XYNS antenna with an arbitrary value of ϕ pointing in an arbitrary direction. The *(ijk)* coordinates of the pointing angle must first be represented in a coordinate system aligned with the local horizon: Up, East, and North *(uen)* are convenient coordinates. A rotation of $-\phi$ around $+j$ is used. Then the *(uen)* coordinates are represented in a reference frame, call it *(abc)*, aligned with the azimuth of the point. A rotation of $-AZ$ around $+u$ brings the point into a reference frame where it is at azimuth $0°$. At this stage an increase in the elevation of the RF axis by a small angle ϵ can be modeled as a simple rotation around $+b$. This produces apparent coordinates $(abc)'$. Then the $(abc)'$ coordinates must be rotated back to $(uen)'$ and then back to $(ijk)'$:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos AZ & -\sin AZ \\ 0 & \sin AZ & \cos AZ \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} \cos\epsilon & 0 & -\sin\epsilon \\ 0 & 1 & 0 \\ \sin\epsilon & 0 & \cos\epsilon \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{46}$$

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos AZ & \sin AZ \\ 0 & -\sin AZ & \cos AZ \end{bmatrix} \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} \tag{47}$$

Equations (45), (46), and (47) can be multiplied out and equation (49) substituted directly to give the complete correction. However the resulting rotation matrix is rather large and not very useful. Instead the small angle approximation is used for ϵ:

$$\begin{bmatrix} i' \\ j' \\ k' \end{bmatrix} \approx \begin{bmatrix} 1 & -\epsilon\sin AZ\cos\phi & -\epsilon\cos AZ \\ \epsilon\sin AZ\cos\phi & 1 & \epsilon\sin AZ\sin\phi \\ \epsilon\cos AZ & -\epsilon\sin AZ\sin\phi & 1 \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \tag{48}$$

The elevation deflection ϵ is assumed to be caused by gravity and represents a lowering of the RF axis. The effect is assumed to be proportional to the force exerted along the current azimuth and perpendicular to the local horizon plane. The component of deflection along increasing elevation angle is:

$$\epsilon = -P_2\cos EL \tag{49}$$

The sign is negative because we initially assumed ϵ was an increase in the elevation. Examining k':

$$\sin Y' \approx \sin Y - P_2(\cos EL\cos AZ\cos Y\cos X - \cos EL\sin AZ\sin\phi\cos Y\sin X) \tag{50}$$

From equations (1) and (10) of the **Coordinate Converions** manual:

$$\cos EL\sin AZ = \cos Y\sin X \quad \textit{and} \quad \cos EL\cos AZ = \cos\phi\sin Y - \sin\phi\cos Y\cos X \tag{51}$$

which we can substitute into equation (50) and yield the approximation:

$$\Delta Y \approx -P_2(\cos\phi \sin Y \cos X - \sin\phi \cos Y) \tag{52}$$

Examining i', we get:

$$\cos Y' \cos X' \approx \cos Y \cos X + P_2(\cos EL \sin AZ \cos\phi \cos Y \sin X + \cos EL \cos AZ \sin Y) \tag{53}$$

Using equation (51) and the derived expression for ΔY from (52), the result is:

$$\Delta X = -P_2 \cos\phi \sin X \sec Y \tag{54}$$

4.7 Other Coordinate Systems

Derivation of the pointing model in the other coordinate systems is most easily handled by noting that equation (3) for AZEL and HADC telescopes is obtained by substituting *-AZ* or *-HA* for X and *EL* or *DC* for Y. The resulting pointing model is the same. Considering equation (1), the sign of ΔX is changed. However, because terms P_1, P_3, P_4, and P_6 are defined in terms of increasing X angles or the location of $X=+90°$, the sign of the rotation used to simulate the effect has flipped as well. For terms P_2 and P_5 the sign of the rotation has not changed, but the terms include a factor of $\sin X$ which does change sign. For equation (2), the sign of ΔY has not changed, nor has the effect of terms P_5, P_7, and P_8. The effect of term P_6 has changed sign, but so has the $\sin X$ factor for that term. The use of equation (51) in the derivation of P_2 and P_8 does not cause a problem because the $\sin X$ term appears as $\sin^2 X$ when substituted into equations (50) and (53). In this way, the model is seen to be correctly defined for left-handed, AZEL, and HADC, as well as right-handed coordinate systems.

It is evident that the model would correctly represent the misalignments even if the XYNS system were rotated in azimuth 90°, *i.e.* for the XYEW system. The only terms whose derivation utilized the azimuth of the fixed axis were P_2 and P_8. Those terms model a gravitational sag, the effect of which is independent of an azimuth rotation of the coordinates.